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ARE ASSET DEMAND FUNCTIONS DETERMINED BY CAPM?

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ABSTRACT

The Capital Asset Pricing Model (CAPM) says that the responsiveness of asset-demands to expected returns depends (inversely) on the variance-covariance matrix of returns, rather than being an arbitrary set of parameters. Previous tests of CAPM have usually computed covariances of returns around sample means, and then checked whether the riskier assets are those with the higher mean returns. We offer a new technique for testing CAPM. The technique requires the use of time series data on actual asset-holdings, and non-linear maximum likelihood estimation. We claim superiority to earlier tests on three grounds. (1) We allow expected returns to vary freely over time. (2) The alternative hypothesis is well-specified: asset-demands are linear functions of expected returns that do not depend on the variance-covariance matrix. (3) The test-statistic has a known distribution; it is simply a likelihood ratio test. We try the technique on yearly data, 1954-1980, for household holdings of a portfolio of six assets: short-term bills and deposits, tangible assets, federal debt, state and local debt, corporate debt, and equities. Our test rejects the CAPM hypothesis.

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1. INTRODUCTION

The Capital Asset Pricing Model (CAPM) provides a compelling framework for modeling the asset demands of investors who care about not only expected returns but risk as well. The model has traditionally been tested empirically by looking for a significant relationship between various assets' expected returns and their risk as measured by covariances with the overall market rate of return.[1] However such tests were in the late 1970s subjected to some powerful methodological criticisms.[2] While more sophisticated tests continue to be proposed,[3] some of the leaders of the finance field appear to believe that the well-specified test of CAPM still does not exist.[4] A well-specified test would consist of a test statistic that has a known distribution under the null hypothesis that CAPM holds, and preferably under some meaningful alternative hypothesis as well. This paper claims to offer such a test.

To understand the test, it might help to think about asset-demands from the macroeconomic viewpoint. Macroeconomists have never doubted that the demands for various assets are functions of their expected returns. The question is, what determines the parameters in these functions? The CAPM answer is that the parameters are inversely related to the coefficient of risk-aversion and the variance-covariance matrix of returns. It is worth recalling that the Tobin-Markowitz model was first developed by a macroeconomist to answer precisely this question,[5] not by a stock market analyst to help choose his clients' portfolios nor by a business school professor to demonstrate the wonders of efficient financial markets. However, empirical tests of CAPM on the aggregate level have at best repeated the methodology used in the microeconomic finance tests.[6]

The test proposed here requires a willingness to use time series data on actual portfolios held, rather than restricting the analysis to time series data on rates of return as most previous studies have done.[7] The data used here are the holdings of the aggregate U.S. household sector among six assets: consumer durables and real estate; short-term U.S. government securities, open market paper, and deposits; long-term federal debt; state and local debt; corporate bonds; and equities. It would be possible, indeed desirable, to apply the technique to more disaggregated data.

2. ESTIMATION OF UNCONSTRAINED ASSET-DEMAND FUNCTIONS

We begin by specifying the general alternative hypothesis within which the CAPM hypothesis is nested. Investors choose their portfolio shares as some linear function of expected one-period returns on the various assets, relative to the expected return on some numeraire asset:

$$x_t = \alpha + \beta(Er_{t+1} - \iota Er^d_{t+1}) \quad , \tag{1}$$

where x_t is a vector of portfolio shares allocated to each of the $G - 1$ assets (the G^{th} asset is eliminated as redundant; in our case $G = 6$ and the redundant asset is Treasury bills and other short-term assets);

Er_{t+1} is a vector of the market's expected one-period real returns on each of the $G - 1$ assets.

Er^d_{t+1} is the market's expected one-period real return on the numeraire asset (in our case, Treasury bills again);

ι is a vector of ones, of length $G - 1$;

α is a vector of $G - 1$ constants; and

β is a $(G - 1) \times (G - 1)$ matrix of coefficients that measures the responsiveness of asset demands to expected returns.

We invert equation (1) to express expected returns as a function of asset shares:[9]

$$Er_{t+1} - \iota Er^{d}_{t+1} = -\beta^{-1}\alpha + \beta^{-1}x_t . \tag{2}$$

A common stumbling block is how to model expectations, which are unobservable. Usually the expected return is assumed constant, and estimated by the sample mean. At best it is allowed to change gradually over time as an ad hoc ARIMA process or as a distributed lag of its own past values, or of other observable variables. But the way we have set up equation (2), all that is needed here is to assume that expectations are rational, i.e. that the ex post realized returns are given by

$$r_{t+1} - \iota r^{d}_{t+1} = Er_{t+1} - \iota Er^{d}_{t+1} + \varepsilon_{t+1} , \tag{3}$$

where the expectational error ε_{t+1} is independent of information I_t available at time t :

$$E(\varepsilon_{t+1}|I_t) = 0 .$$

(For our purposes it is necessary only that I_t include x_t . But it can contain other variables as well.)

From (2) and (3) we have

$$r_{t+1} - \iota r^{d}_{t+1} = -\beta^{-1}\alpha + \beta^{-1}x_t + \varepsilon_{t+1} . \tag{4}$$

Two aspects of equation (4) are noteworthy. All variables are observable. And by the rational expectations assumption, the error term is independent of x_t . This means that we can use equation-by-equation Ordinary Least Squares (OLS) to estimate the constant terms and β^{-1} .[9] An attractive property of the

Table 1: Unconstrained Estimation of Inverted Asset-Demand Function

Equation-by-equation OLS. Sample: 1954-1980

Dependent variable		β^{-1}: coefficients on shares of portfolios allocated to:									
Real rate of return on asset relative to short-term bills:	Constant	Tangible assets	Long-term federal debt	State and local debt	Corporate bonds	Equities	D.W.	SSR	R^2	log likelihood	F(5,21)
Tangible assets	-.103 (.409)	.251 (.594)	.279 (.562)	-4.755 (2.750)	3.005 (2.838)	.003 (.445)	2.06	.00648	.52	74.21	4.62*
Long-term federal debt	1.15 (1.19)	-2.071 (1.736)	-2.260 (1.642)	-7.756 (8.040)	22.329* (8.299)	-1.711 (1.300)	2.18	.005538	.52	45.25	4.51*
State and local debt	.979 (1.637)	-2.090 (2.378)	-5.163* (2.250)	-17.020 (11.017)	40.202* (11.372)	-1.635 (1.782)	1.70	.10399	.54	36.74	5.01*
Corporate bonds	.780 (.931)	-1.565 (1.352)	-3.038* (1.279)	-14.240* (6.262)	30.394* (6.464)	-1.365 (1.013)	1.97	.03359	.67	51.99	8.45*
Equities	.110 (2.600)	-.612 (3.776)	6.412 (3.573)	-19.562 (17.491)	3.991 (18.054)	.500 (2.828)	2.08	.26209	.32	24.26	1.95

β^{-1}	$-(G-1)\frac{T}{2}\log 2\pi$	$-\frac{T}{2}\log\lvert\Omega\rvert$	$-T(G-1)/2$	= log likelihood
unconstrained	-124.06	450.98	-67.50	259.42*
constrained to 0	-124.06	412.49	-67.50	220.93

*Significant at the 95% level. (Standard errors in parentheses.)

specification of equation (4) is that it allows expected returns to vary from period to period as much as they want. Fluctuation in interest rates, the expected inflation rate, the expected rate of return on equity, etc., has indeed been large in recent years. Furthermore we have made no ad hoc assumptions about what determines actual returns or expected returns, other than that expectations are rational.

Table 1 reports the results of the OLS estimation. The estimates indicate, for example, that it would take a 30.39 per cent increase in the expected annual return on corporate debt to induce investors to accept an increase in their holdings of corporate debt equal to 1 per cent of their portfolio. This assumes that the increase comes at the expense of the omitted asset, short-term bills and deposits. To calculate the effect of a 1 per cent increase in corporate debt at the expense of another asset, take the difference of the two relevant coefficients.

Only a few of the coefficients appear significantly different from zero by t-tests. But all but one of the individual equations do appear significant by F-tests.[10] To do an overall test of the system of equations we must compare the log likelihood when the coefficients are constrained to zero, to the likelihood unconstrained.[11] The numbers are 220.93 and 259.42 , respectively. Since twice the difference is distributed χ^2 , the reduction in the likelihood that would result from the constraint that the coefficients are all zero is highly significant.

3. ESTIMATION OF CONSTRAINED ASSET-DEMAND FUNCTIONS

We now consider the restrictions imposed on the asset-demand function (1) by CAPM. Since the econometrics are necessarily discrete-time, we adopt a discrete-time theoretical framework. Consider four assumptions:

(A1) perfect capital markets

(A2) optimization of end-of-period expected utility

(A3) normal distribution of returns

(A4) constant relative risk-aversion.[12]

As we show in Appendix 1, these assumptions imply a restriction on the asset-demand function (1) that is astonishingly simple:[13]

$$\beta = [\rho\Omega]^{-1}$$

where ρ is the constant of relative risk-aversion and Ω is the $G-1 \times G-1$ variance-covariance matrix of returns. Intuitively, investors will respond less to a given disparity in expected returns if the perceived uncertainty (Ω) is high, or if their risk-aversion (ρ) is high.

The conventional way to estimate the optimal portfolio is to estimate the sample variance-covariance matrix of ex post returns. But such an approach presumes that expected returns are constant, an assumption we have been trying to avoid, and on the other end leaves us with a constant estimated optimal portfolio, which would be difficult to compare rigorously to the time-varying actual portfolio x_t. The key insight of this paper is that Ω is precisely the variance-covariance matrix $E\varepsilon\varepsilon'$ of the error term in equation (4), and that the equation should be estimated subject to this constraint. The imposition of a constraint between the coefficient matrix and the error variance-covariance matrix is unusual in econometrics, and requires maximum likelihood estimation (MLE). Once we have done this estimation, we have our test of CAPM: we compare the log likelihood at the constrained maximum, to the log likelihood of the unconstrained version that we have already done in Table 1. Appendix (2) shows the constrained likelihood function and its derivatives, and describes the program used to maximize it.

If the aim were to assume CAPM a priori and to use the information to get the most efficient possible estimates of the parameters, then one might wish to impose not only the constraint that the coefficient matrix is proportional to the variance-covariance matrix Ω , but to impose as well an a priori value for the constant of proportionality, which is the coefficient of relative risk-aversion ρ . Friend and Blume (1975) offer evidence that ρ may be in the neighborhood of 2.0 . We report in Table 2 the parameter estimates for the case $\rho = 2.0$. The results look quite different from those in Table 1. If one believes the constraints, then the difference is simply the result of more efficient estimates. One has to invert the matrix in order to recover the original β matrix and see which assets are close substitutes for which other assets. These coefficients are reported in Table 3. We can infer from the negative numbers in the fourth row for example that corporate bonds are substitutes for federal debt, state and local debt, and equities.[14]

But we have chosen in this paper to emphasize the use of our technique to test the CAPM hypothesis, rather than the use of the technique to impose the hypothesis. The log likelihood for the estimates in Table 2 is 154.19 , a substantial decrease from the unconstrained log likelihood 259.42 . In other words, the fit has worsened. Twice the difference is far above the 5 per cent critical level. This constitutes a clear rejction of the CAPM hypothesis.

Perhaps the constraint that $\rho = 2.0$ is too restrictive and accounts for the magnitude of the decline in the likelihood function. We searched over the range $\rho = 1.0$ to $\rho = 20.0$ with the technique. The likelihood function

Table 2: Constrained Estimation of Inverted Asset-Demand Function

MLE. Sample: 1954-1980

ρ constrained to 2.0

Dependent variable Real rate of return on asset relative to short-term bills:	Constant	β^{-1}: coefficients (constrained to $\rho\Omega$) on shares of portfolios allocated to: Tangible assets	Long-term federal debt	State and local debt	Corporate bonds	Equities
Tangible assets	.013	.02087				
	(.137)	(.01177)		(symmetric)		
Long-term federal debt	.020	.00005	.00961			
	(.040)	(.00540)	(.00719)			
State and local debt	-.031	-.00186	-.00070	.00678		
	(.041)	(.02478)	(.00312)	(.00842)		
Corporate bonds	-.005	-.00027	.00029	.00078	.01344	
	(.066)	(.01329)	(.00638)	(.00604)	(.00578)	
Equities	-.018	.00256	.00303	.00360	.00104	.02004
	(.085)	(.00903)	(.01042)	(.00522)	(.00652)	(.00988)

$-(G-1)\frac{T}{2}\log 2\pi$	$-\frac{T}{2}[\log\|\Omega\| + G-1]$	= log likelihood
-124.06	278.25	154.19

Table 3: Constrained Estimate of Pre-inverted Asset-Demand Function

β^{-1} in Table 2 inverted = $(\rho\Omega)^{-1}$

ρ constrained to 2.0

The demand for the assets listed below	depends on the expected real return (relative to the real return on bills) of the following assets				
	Tangible assets	Long-term federal debt	State and local debt	Corporate bonds	Equities
Tangible Assets	51.04	4.63	20.23	.59	-10.88
Long-term federal debt	4.63	112.82	24.94	-2.08	-22.04
State and local debt	20.23	24.82	176.57	-7.47	-37.73
Corporate bonds	.59	-2.08	-7.47	75.05	-2.31
Equities	-10.88	-22.04	-37.73	-2.31	61.53
Short-term bills and deposits (= - sum of other rows)	-65.61	-118.27	-176.54	-63.78	+11.43

increases with ρ in this range, but at $\rho = 20.0$ the log likelihood was still only 169.68 , which is again a clear rejection of the CAPM hypothesis. There did not seem to be any point in searching beyond this already implausibly high range.

4. CONCLUSION

How could CAPM fail to hold? Do our results imply that investors are irrational? The failure of any one of the four CAPM assumptions listed above could explain the finding. Investors may be rational but may have to optimize subject to constraints such as imperfect capital markets. Or they may be maximizing an intertemporal utility function, à la Merton (1973) and Breeden (1979), that is more complicated than a function of the mean and variance of end-of-period real wealth. Or returns may not be normally distributed. Or investors may not have a constant coefficient of relative risk-aversion.

Our rejection of the null hypothesis could also be due to the failure of other assumptions that we have made in our model, but that are not part of CAPM most narrowly defined: homogeneous investors, a constant variance-covariance matrix, rational expectations, the aggregation of the assets into six, and the accurate measurement of the holdings of those assets. The test could be refined with respect to most of these assumptions, especially by greater disaggregation of the assets or the holders.

The Capital Asset Pricing Model is a very attractive way to bring structure to asset-demand functions. One possibility is that true asset demands are equal to those given by the CAPM formula <u>plus</u> some other factors. The other factors would not necessarily have to be large for our technique to reject the null hypothesis. This is entirely appropriate. We are testing the hypothesis that CAPM holds exactly. But it does allow the possibility that CAPM may still have something to tell us about asset demands despite our statistical rejection of it.

Appendix 1

In this appendix we derive the correct form for the asset-demands of an investor who maximizes a function of the mean and variance of his end-of-period real wealth.

Let W_t be real wealth. The investor must choose the vector of portfolio shares x_t that he wishes to allocate to the various assets. End-of-period real wealth will be given by:

$$\begin{aligned} W_{t+1} &= W_t + W_t x_t' r_{t+1} + W_t(1-x_t'\iota)r^d_{t+1} \\ &= W_t[x_t' z_{t+1} + 1 + r^d_{t+1}] , \end{aligned} \tag{A1}$$

where we have defined the vector of returns on the $G - 1$ assets relative to the numeraire asset (deposits): $z_{t+1} \equiv r_{t+1} - r^d_{t+1}$.

The expected value and variance of end-of-period wealth (5), conditional on current information, are as follows:

$$\begin{aligned} EW_{t+1} &= W_t[x_t' Ez_{t+1} + 1 + Er^d_{t+1}] \\ VW_{t+1} &= W_t^2[x_t'\Omega x_t + Vr^d_{t+1} + 2x_t' \operatorname{Cov}(z_{t+1}, r^d_{t+1})] , \end{aligned}$$

where we have defined the variance-covariance matrix of relative returns:

$$\Omega \equiv E(z_{t+1} - Ez_{t+1})(z_{t+1} - Ez_{t+1})' .$$

The hypothesis is that investors maximize a function of the expected value and variance:

$$F[E(W_{t+1}), V(W_{t+1})] .$$

We differentiate with respect to x_t :

$$\frac{dF}{dx_t} = F_1 \frac{dEW_{t+1}}{dx_t} + F_2 \frac{dVW_{t+1}}{dx_t} = 0 .$$

$$F_1 W_t [Ez_{t+1}] + F_2 W_t^2 [2\Omega x_t + 2 \operatorname{Cov}(z_{t+1}, r^d_{t+1})] = 0 .$$

We define the coefficient of relative risk-aversion $\rho \equiv -W_t 2F_2/F_1$, which is assumed constant. Then we have our result:

$$Ez_{t+1} = \rho \operatorname{Cov}(z_{t+1}, \iota r^d_{t+1}) + \rho\Omega x_t . \tag{A2}$$

This is just equation (2) with β^{-1} constrained to be $\rho\Omega$, as claimed by equation (5) in the text. (There is also a constraint imposed on the intercept term α . But it is inconvenient to impose this constraint in the econometrics. Nor do we need it, since the constraint on the coefficient matrix already gives us 25 overidentifying restrictions.)

For economic intuition, we can invert (A2) to solve for the portfolio shares, the form analogous to (1):

$$x_t = -\Omega^{-1} \operatorname{Cov}(z_{t+1}, r^d_{t+1}) + (\rho\Omega)^{-1} Ez_{t+1} . \tag{A3}$$

The asset demands consist of two parts. The first term represents the "minimum-variance" portfolio, which the investor will hold if he is extremely risk-averse $(\rho = \infty)$. For example, suppose he views deposits as a safe asset, which requires that the inflation rate is nonstochastic. Then his minimum-variance portfolio is entirely in deposits: the $G - 1$ entries in x_t are all zero because the Cov in (A3) is zero. The second term represents the "speculative" portfolio. A higher expected return on a given asset induces investors to hold more of that asset than is in the minimum-variance portfolio, to an extent limited only by the degree of risk-aversion and the uncertainty of the return.

APPENDIX 2

Using the assumption of normally-distributed returns, the log likelihood function when no constraint is imposed on the coefficient matrix is

$$L = -\frac{(G-1)T}{2}\log 2\pi = \frac{T}{2}\log|\Omega| - \frac{1}{2}\sum_{t=1}^{T} \varepsilon'_{t+1}\Omega^{-1}\varepsilon_{t+1}\ , \tag{6}$$

where we know from equation (4) that $\varepsilon_{t+1} = (r_{t+1} - \iota r^d_{t+1}) - c - \beta^{-1}x_t$. The unconstrained MLE is simply the OLS estimates that we already looked at in Table 1.

For the constrained MLE, we substitute $\rho\Omega$ for β^{-1} . Ω now appears in the likelihood function in two ways. To maximize, we differentiate. The derivatives with respect to the coefficient of risk-aversion and the intercept term are easy:

$$\begin{aligned}
\partial L/\partial\rho &= -\sum \varepsilon'_{t+1}\Omega^{-1}(\partial\varepsilon_{t+1}/\partial\rho) \\
&= -\sum \varepsilon'_{t+1}\Omega^{-1}(-\Omega x_t) \\
&= \sum \varepsilon'_{t+1}x_t \\
\partial L/\partial c &= -\sum \varepsilon'_{t+1}\Omega^{-1}(\partial\varepsilon_{t+1}/\partial c) \\
&= \sum \varepsilon'_{t+1}\Omega^{-1}\ .
\end{aligned}$$

The derivative with respect to the elements of the variance-covariance matrix is trickier. It will help to perform the Choleski factorization of the matrix:

$$S'S = \Omega\ ,$$

where S is a lower triangular matrix, and to differentiate with respect to the elements of S, which can be thought of as the $G - 1$ generalization of the standard deviation. We first use the two facts (from Theil (1971, pp. 31-32), equations (6-14) and (6-8), respectively):

$$\frac{\partial \det|\Omega|}{\partial \Omega} = \Omega^{-1} \quad \text{and} \quad \frac{\partial(\varepsilon'\Omega^{-1}\varepsilon)}{\partial \Omega^{-1}} = \varepsilon\varepsilon' .$$

$$\begin{aligned} \partial L/\partial\Omega &= -\frac{T}{2}\,\Omega^{-1} - \frac{1}{2}\sum\,[-\Omega^{-1}\varepsilon_{t+1}\varepsilon'_{t+1}\Omega^{-1} + 2\Omega^{-1}(\partial\varepsilon_{t+1}/\partial\Omega)] \\ &= -\frac{T}{2}\,\Omega^{-1} + \frac{1}{2}\sum\,[\Omega^{-1}\varepsilon_{t+1}\varepsilon'_{t+1}\Omega^{-1} + 2\rho\Omega^{-1}x_t] . \end{aligned}$$

Then we use the chain rule.

$$\begin{aligned} \partial L/\partial S &= (\partial L/\partial\Omega)(\partial\Omega/\partial S) \\ &= \{-\frac{T}{2}\,(S'S)^{-1} + \frac{1}{2}\sum\,[(S'S)^{-1}\varepsilon_{t+1}\varepsilon'_{t+1}(S'S)^{-1} + 2\rho(S'S)^{-1}x_t]\}2S . \end{aligned}$$

Setting the derivatives equal to zero gives first order conditions that characterize the MLE. However, due to nonlinearity they cannot be solved explicitly for the estimates of ρ , c , and Ω or S . The Berndt, Hall, Hall and Hausman (1974) algorithm uses the first derivatives to find the maximum of the likelihood function in non-linear models. For our problem, we modified a program written by Paul Ruud, based on this algorithm.[15]

APPENDIX 3

DATA

The main source for data on supplies of nine assets held by households was the Federal Reserve Board's Balance Sheets for the U.S. Economy (October 1981) Table 702. This source was used in place of the Fed's Flow of Funds Accounts, Assets and Liabilities Outstanding, to which it is closely related, because only the Balance Sheets include data for tangible assets, i.e. real estate and consumer durables (see page iii of the Flow of Funds for an explanation). The variables used in the econometrics are shares of wealth, the supply of the asset in question divided by the sum of all nine asset supplies.

The asset supplies were taken from the Balance Sheets as follows. Real estate is line 1 (total tangible assets) minus line 7 (consumer durables).[16] Consumer durables is line 7.[17] Open market paper is line 25. Short-term U.S. government securities are line 20 [not available before 1951]. Deposits is the sum of lines 13, checkable deposits and currency, 14, small time and savings deposits, 15, money market fund shares, and 16, large time deposits. Long-term federal debt is line 18 (U.S. government securities) minus line 20. State and local debt is line 23. Private bonds are line 24 (corporate and foreign bonds) plus line 26 (mortgages held).[18] Finally, equities are line 27 (corporate equities) plus line 32 (noncorporate business equity).[19]

For three of the asset supplies--long-term federal debt, state and local bonds, and private bonds--the numbers represent book value and must be multiplied by some measure of current market prices to get the correct measure of market value. The very large decline in prices of bonds over the postwar period make this correction a crucial one. (Equities and tangible assets are already measured at market value, while capital gains and losses are irrelevant for the three short-term assets.) Measures of the current market bond prices are reported by Standard and Poor's Trade and Security Statistics Security Price Index Record (1982): page 235 for U.S. government bond prices, 233 for municipal bond prices, and 231 for high grade corporate bond prices. Standard and Poor's computes the price indexes from yield data, assuming a 3% coupon with 15 years to maturity for the federal bonds and a 4% coupon with 20 years to maturity for the other two.[20]

Among the rates of return, the two most problematical are those on real estate and durables, taken here as the percentage change in price indices reported in the Economic Report of the President 1982: the home purchase component of the CPI (p. 292) and the durable goods personal consumption expenditure component of the GNP deflator (p. 236). There exist better measures of house prices, and unpublished estimates of imputed service returns on housing and durables, but they are not available for the entire sample period. When the two tangibles are aggregated, we use real estate appreciation as the return.

The short-term assets are straight-forward. The rate of return on open market paper is the interest rate on commercial paper from the Federal Reserve Board: Banking and Monetary Statistics 1941-1970, table 12.5, Annual Statistical Digest 1970-79, table 22A, and ASD 1980, table 25A. The rate of return on short-term government securities is the treasury bill rate: 9-12 month issues (certificates of indebtedness and selected note and bond issues; the 1-year bill market yield rate is not available before 1960) from BMS 1941-1970, and the 1-year bill secondary market from ASD 1970-1979, table 22A, and ASD 1980, table 25A. The rate of return on deposits is the rate on 90-day bankers' acceptances from BMS 1941-1970, table 12.5, ASD 1970-1979, table 22A, and ASD 1980, table 25A. Alternatives such as the return on money market funds might be theoretically preferable but are not available for the early part of the sample period. Note that in aggregating non-interest paying money together with interest-paying accounts, we are assuming that the former performs an implicit liquidity service that brings its return up to the explicit return of the latter. When the three short-term assets are aggregated, we use the Treasury bill rate as the return.

Each of the long-term assets entails a yield plus capital gains. For each of the three kinds of bonds, capital gains are percentage change in the same bond prices from Standard and Poor's Trade and Securities Statistics that were discussed above. The yields are from the same source: respectively, the median yield to maturity of a number of government bonds restricted to those issues with more than ten years to maturity, p. 234, an arithmetic average of the yield to maturity of fifteen high grade municipal bonds, p. 232, and an average of the AAA Industrial and Utility bonds, p. 219. (The yields are also available from the Fed sources: BMS 1941-1970, table 12.12, ASD 1970-1979, table 22A and ASD 1980, table 25A.) For equities, capital gains are percentage change in Stanford and Poor's index of common stock prices from BMS 1941-1970, table 12.16, ASD 1970-1979, table 22A, and ASD 1980, table 26A. To capital gains we add the dividend price ratio on common stock, from BMS 1941-1970, table 12.19, ASD 1970-79, table 22A, and ASD 1980, table 25A.

The foregoing are all nominal returns. To convert to real returns we use the percentage change in the CPI, from the Economic Report of the President 1982. To be precise we divide one plus the nominal return by one plus the inflation rate. Subtracting the inflation rate from the nominal return would give approximately the same answer, and when we computed real returns relative to the numeraire asset the two inflation rates would conveniently drop out, but this answer would differ from the correct one by a convexity term.

Absent from the calculations is any allowance for differences in tax treatment. In particular, the returns on state and local bonds, and to some extent on tangibles, are here understated relative to the other assets because they are tax-free. The unconstrained constant term that we allow for in the econometrics should capture most of this effect (and any other constant omitted factors such as the service return from tangibles, as well). But it would be desirable to compute after-tax real returns instead.

FOOTNOTES

1. Two common references are Black, Jensen and Scholes (1972) and Blume and Friend (1973).
2. See Roll (1977) and Ross (1978).
3. For example, Gibbons (1982).
4. E.g. Ross (1980.
5. Tobin (1958).
6. Nordhaus and Durlauf (1982) is one of the very few attempts to test CAPM on a comprehensive portfolio of highly aggregated assets, similar to the portfolio used in the present study: corporate fixed capital, housing, short-term government bonds, long-term government bonds, and consumer durables.
7. A few studies, such as Friend and Blume (1975) have dared to look at actual portfolios held by households, but not in time series form. Use of consumption data in tests of Breeden's (1979) intertemporal CAPM may have accustomed the finance profession to time series data on quantity.
8. The choice to express returns relative to a numeraire is not restrictive. We could generalize (1) slightly to

$$x_t = \alpha + \tilde{\beta} E \begin{bmatrix} r_{t+1} \\ \cdots \\ r^d_{t+1} \end{bmatrix}$$

where $\tilde{\beta}$ is $G - 1$ by G. Then when we invert

$$E \begin{bmatrix} r_{t+1} \\ \cdots \\ r^d_{t+1} \end{bmatrix} = -\tilde{\beta}^{-1}\alpha + \tilde{\beta}^{-1}\alpha \ ,$$

we need only subtract the last row from each of the others to get an equation of the precise form as (2). In what follows we only use (2) anyway.

Note, incidentally, that we must avoid the temptation to think that because "expected inflation cancels out" relative real returns can be replaced with relative nominal returns. If i^j_{t+1} is the nominal return on a particular asset j and π_{t+1} is the inflation rate,

$$Er^j_{t+1} \equiv E \frac{1+i^j_{t+1}}{1+\pi_{t+1}} \neq 1 + Ei^j_{t+1} - E\pi_{t+1} \ .$$

9. The validity of the technique depends on the assumption that the asset-demand function (1) holds exactly. If asset demands are determined by CAPM plus other factors, the null hypothesis does not hold.
10. The test that the coefficients in a row are significantly different from zero is a test that the asset in question is not a perfect substitute for Treasury bills and other short-term assets. The 5 per cent critical level for the F statistic is 2.68 .
11. The estimated log likelihood is given by equation (6) in Appendix 2, with the estimates $\hat{\varepsilon}_t$ and $\hat{\Omega}$ substituted in for the true parameters. The last of the three terms is simply $-\frac{(G-1)T}{2}$, because $\hat{\varepsilon}_t'\hat{\Omega}^{-1}\hat{\varepsilon}_t =$ tr $\hat{\varepsilon}_t'\hat{\Omega}^{-1}\hat{\varepsilon}_t =$ tr $\hat{\varepsilon}_t\hat{\varepsilon}_t'\hat{\Omega}^{-1} =$ tr $\hat{\Omega}\hat{\Omega}^{-1} = G - 1$. (See G. S. Maddala, Econometrics (N.Y.: McGraw-Hill) 1977, p. 487 after equation C-50.) So the test statistic varies only with the determinant of $\hat{\Omega}$. Under the zero-

coefficient constraint, $\hat{\Omega}$ is simply the variance-covariance matrix of the raw data, the relative rates of return. (We do allow for a non-zero constant term.) Unconstrained, $\hat{\Omega}$ is the variance-covariance matrix of the residuals of the $G - 1$ equations. Because the residuals are correlated across equations, $T \log|\hat{\Omega}|$ is somewhat less than the sum of the logs of the $G - 1$ individual equations' sums of squared residuals, and the log likelihood is correspondingly greater than the sum of the $G - 1$ individual log likelihoods. (The 5% significance level for the χ^2 test is 37.65) .

12. The utility function will have a constant coefficient of relative risk-aversion if it is a power function:

$$u(w) = \frac{1}{1-\rho} w^{1-\rho} .$$

We could replace the last two assumptions with the single assumption of quadratic utility. But that assumption is unrealistic, and we will need to assume a normal distribution anyway in order to do our maximum likelihood estimation.

The solution to the one-period maximization problem considered here (Assumption 2) will give the same answer as the general intertemporal maximization problem if the utility function is further restricted to the logarithmic form, the limiting case as ρ goes to 1.0 , or if expected returns in future periods are independent of the realization of this period's return. See Merton (1973, pp. 877-78) or Fama (1970).

13. The derivation is relegated to the Appendix, not because of any degree of complexity, but rather because of its familiarity. Some similar formulations are Friend and Blume (1975, equation 5), Black (1976, equation 4), and Friedman and Roley (1979, equation 20').

14. On the other hand, state and local bonds, surprisingly, appear to be complementary to federal debt, as is reflected also by a negative sign in the corresponding entry in Table 2. This illustrates how big a difference it makes to compute the covariance around a time-varying expected return (half of -.0007 in this case) rather than the simple covariance around the mean, which does turn out to be positive (+.0052 in this case) as we would expect. If the model is correct, the apparent positive correlation of real returns on these two kinds of bonds was a positive correlation of their *expected* returns, not the unexpected returns.

15. This is the same program used in Frankel and Engel (1982). An analytic solution was derived in Frankel (1982) for a problem that was the same but for the absence of an intercept term to be estimated.

16. An alternative here is to subtract lines 38 and 39, mortgages owed by households, viewing them as a liability that is institutionally tied to the real estate asset. One cannot explain otherwise households' decision to hold on net a negative quantity of mortgages on risk-return considerations, as the mortgage rate is higher than that on other bonds.

17. An alternative here is to subtract lines 40 and 41, consumer credit, viewing it as a liability that is tied to the durables asset, for the same reason as in the previous footnote.

18. An alternative here is to add in also lines 30 (life insurance reserves), 31 (pension fund reserves) and 34 (miscellaneous assets). These cannot be treated as separate assets because their rates of return are not

available, but it is desirable to have all forms of wealth included somewhere, and they fit into the category of private bonds better than anywhere else.

19. An alternative here is to subtract the difference of lines 44 and 33, representing net security credit, viewing it as a liability that is tied to the equity asset.

20. These same bond prices were reported in the Federal Reserve Board's Banking and Monetary Statistics 1941-1970. They have been discontinued apparently because the Capital Markets Section at the Federal Reserve Board feels that dispersion in the coupon rate and shifts in the term structure make the aggregation of all long-term bonds no longer possible. But some correction for the market price is clearly preferable to none.

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